

# Pseudo-telepathy games and genuine NS $n$ -way nonlocality using graph states

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## Abstract

We define a family of pseudo-telepathy games using graph states that extends the Mermin games. This family also contains a game used to define a quantum probability distribution that cannot be simulated by any number of PR boxes. We extend this result proving that the probability distribution obtained by the Paley graph state on 13 vertices cannot be simulated by any number of 4-partite non local boxes and that the Paley graph states on more than  $n^2 2^{2n-2}$  vertices provide a probability distribution that cannot be simulated by  $n$ -partite nonlocal boxes .

## 1 Introduction

Quantum non-locality is one of those rare physical phenomena that are discovered to be deeply rooted in the foundations of physics long before they are properly understood and universally accepted. Originally used by Einstein, Podolsky and Rosen [15] in 1935 in their attempt to prove quantum mechanics incomplete, it was given a completely new avatar by John S Bell in his seminal work [3] of 1964 and has now taken the form of a physical principle thanks to remarkable results like no-communication theorem [23] and provided interesting mathematical tools like non-signaling boxes [25] etc.

An important field of interest in quantum information theory has been the quantification of this non-locality. It is clear that physics is non-local, but what can be said about how non-local is a physical system? There have been two major approaches to this problem. The first approach is to consider the cost of simulating characteristics of a physical system (more precisely, probability distributions exhibited by the physical system) with non-signaling boxes [13], one way communication between observers [29], bounded communication in the average or the worst case scenario [7], etc.

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The second approach to defining a measure of non-locality is the amount by which a non-signalling probability distribution (exhibited by a non-signalling box) violates Bell type inequalities such as the CHSH inequalities [14]. CHSH inequalities have been extended to the multipartite scenario by Svetlichny [28, 27] and weaker inequalities proposed by Pironio et al. [2] that, within their notion of ‘ $n$ -way nonlocality’, strengthen our intuitive understanding of multipartite non-local nature of certain quantum correlations. Another concept of ‘genuinely  $n$ -way NS non-local’, as introduced in [2], corresponds to those probability distributions that cannot be simulated by  $n$ -partite non-signalling boxes and plays central role in our article.

At present, only bi-partite non-signalling boxes are well understood. A classification of extremal points of the non signaling probability distribution polytope in tripartite scenario has been done recently in detail in [24], but a more intuitive understanding is still required.

Apart from Bell’s inequalities, one approach to describe the non-classical nature of quantum mechanics is using pseudo-telepathy games. Pseudo-telepathy games aim at providing a simple and natural interpretation of quantum non-locality. They have been vividly described in Brassard et al. [8], as protocols that can play important role in experimental verification of non-local nature of our world, in some cases even when measurement detectors are considerably inefficient [11]. On the theoretical side, they also provide an interesting measure of non-locality, in terms of probability by which the best strategy of a classical player can win the games.

In present article, we extend an important result in [1] that uses only the property of non-signalling to present a quantum correlation that cannot be simulated by bi-partite PR Boxes. We show that there exist quantum probability distributions that cannot be simulated by  $n$ -partite non-signalling boxes, for any given  $n$ , without requiring any knowledge of the detailed nature of extremal points of non-signalling probability distribution polytope in multipartite setting.

To achieve this goal, we present a family of pseudo-telepathy games using graph states. If players share a graph state  $|G\rangle$  then using a simple protocol consisting of measurements in the diagonal basis when the input is 1 and in the standard basis when the input is 0, they win perfectly the game defined on  $G$ . These games generalize a well known game called Mermin’s parity game, originally described as a 3-player game in [18] and studied as a general  $n$ -player game in [9]. Moreover, the correlations considered in [1] (and introduced in [10]), that cannot be simulated using PR boxes, are the ones obtained in the games using the graph  $C_5$  (cycle on five vertices).

Using the fact that  $C_5$  is a special case of a more general family called Paley graphs, we show that the probability distribution obtained by the quantum strategy on Paley graph states on more than 5 vertices cannot be simulated by tri-partite non-signalling boxes and that four-partite non-signalling boxes cannot simulate the probability distribution obtained in the game on the Paley graph state on 13 qubits.

Finally, using a graph theoretic property called existential closure [5, 12], we infer that the probability distributions obtained with the Paley graph states on greater than  $n^2 2^{2n-2}$  qubits cannot be simulated by  $n$ -partite nonlocal boxes.

## 2 Pseudo-telepathy graph games

Given  $n$  players that are not allowed to communicate, each of them receives an input  $x_i$  and is asked to provide an output  $a_i$ . Let  $I_i$  ( $O_i$ ) be the domain of the inputs (outputs) of the player  $i$  and  $I \subseteq I_1 \times \dots \times I_n$  ( $O = O_1 \times \dots \times O_n$ ) the input (output) domain. A game  $G_m$  is defined by a relation  $\mathcal{L}(G_m) \subseteq I \times O$  representing the set of loosing configurations *i.e.* if the players are asked questions  $x_1, \dots, x_n$  and they answer  $a_1, \dots, a_n$  with  $x_1, \dots, x_n, a_1, \dots, a_n \in \mathcal{L}(G_m)$  the players loose the game.

Note that in general the set of legitimate questions  $I$  is a subset of  $I_1 \times \dots \times I_n$ . When  $I = I_1 \times \dots \times I_n$  we say that the game is without promise (defining the game with a loosing condition makes it easier to extend promise games to general games : the extra inputs have no loosing condition associated to them).

In the following, we will consider only the case where the inputs and outputs are in  $\{0, 1\}$ .

Given a graph  $G = (V, E)$ , for any subset of vertices  $D$ , we define its odd and even neighborhood as follows:  $Even(D) = \{v \in V, |N(v) \cap D| = 0 \bmod 2\}$ , and  $Odd(D) = \{v \in V, |N(v) \cap D| = 1 \bmod 2\}$ , where  $N(v)$  is the neighborhood of  $v$ . We define graph games as follows : we identify the players with the vertices of the graph. The players loose if and only if there exists a eulerian induced subgraph  $D$ , for which when 1 is asked to the players in  $D$  and 0 to the players in  $Odd(D)$ , the sum of the answers modulo 2 of the players in  $D \cup Odd(D)$  is different from the number of edges of the subgraph induced by  $D$ .

**Definition 1. Graph game  $G_G$  :** Given a graph  $G = (V, E)$  on  $n$  vertices, we define the loosing set of a graph game  $G_G$  as follows:  $x_1, \dots, x_n, a_1, \dots, a_n$  is in  $\mathcal{L}(G_G)$  if and only if  $\exists D \subseteq V$  such that

- $D \subseteq Even(D)$
- $x_k = 1$  if  $k \in D$  and 0 if  $k \in Odd(D)$
- $\sum_{i \in D \cup Odd(D)} a_i = |E(D)| + 1 \bmod 2$  where  $E(D)$  is the set of edges of the subgraph induced by  $D$ .

For any  $D \subseteq Even(D)$ , let  $Q(D)$  be the set of questions for which  $x_i = 1$  if  $i \in D$  and  $x_i = 0$  if  $i \in Odd(D)$ .

First, we provide a condition on the graph  $G$  ensuring that if each player uses a classical deterministic strategy they cannot avoid the loosing configurations of the game  $G_G$ .

**Lemma 1.** Given  $G = (V, E)$  the graph game  $G_G$  cannot be won perfectly by any classical deterministic strategy, if there exists a  $D \subseteq V$  such that  $D \subseteq Even(D)$  and  $|E(D)| = 1 \bmod 2$  where  $E(D)$  is the set of edges of the subgraph induced by  $D$ .

*Proof.* Given a subset  $U$  of  $V$ , such that  $|E(U)| = 1 \bmod 2$ , let  $u_1, u_2, \dots, u_k$  be vertices in  $U$ , with  $k = |U|$ . Suppose that there exists a deterministic strategy that never loses for

the game  $G_G$ , and let  $a_v^0$  be the output of the player  $v$  when his input is 0 and  $a_v^1$  if his input is 1.

When  $D = \{u_i\}$ ,  $Odd(D) = N(u_i)$  and  $D \subseteq Even(D)$ , thus if the players never lose then:

$$a_{u_i}^1 + \sum_{j \in N(u_i)} a_j^0 = 0 \mod 2 \quad (1)$$

and when  $D = U$

$$\sum_{l=1}^k a_{u_l}^1 + \sum_{j \in Odd(U)} a_j^0 = 1 \mod 2 \quad (2)$$

Adding equations 1 for  $i$  from 1 to  $k$  gives  $\sum_{l=1}^k a_{u_l}^1 + \sum_{j \in Odd(U)} a_j^0 = 0 \mod 2$  which contradicts with equation 2.  $\square$

Lemma 1 directly implies (see [8]) that even with a probabilistic strategy and with shared random variables classical players have a non zero probability to loose the game  $G_G$  if  $G$  has an Eulerian induced subgraph with an odd number of edges. (A simple example of such subsets are induced odd cycles.)

However, for any graph game  $G_G$ , if the players share the graph state  $|G\rangle$ , there exists a strategy which ensures that they never loose.

**Lemma 2.** *There is a strategy that allows quantum players to never loose in the game  $G_G$ .*

*Proof.* If the players share the quantum state  $|G\rangle$ , and for input 1(0), they measure  $X(Z)$ , then the output completely avoids the losing conditions  $\mathcal{L}(\mathcal{G})(G_G)$  on this game.

Given a graph state, if  $(-1)^s \prod_{i \in V} \sigma_i^k$  is in the stabilizer, where  $\sigma_i^k$  are Pauli matrices, then qubit  $i$  is measured in the  $\sigma_i^k$  basis and gives the classical result  $m_i \in \{0, 1\}$  (0 corresponds to projector  $(I + \sigma)/2$  and 1 to  $(I - \sigma)/2$ ), the measurement results satisfy following relation:  $\sum_i m_i = s \mod 2$ .

Now we show that, given any  $D \subseteq V$  such that  $D \subseteq Even(D)$ , there is an operator in the stabilizer that corresponds to  $s = |E(D)|$ . Label the vertices in  $V$  with integers  $1, 2, \dots, |V|$  such that vertices in  $D$  are assigned the first  $|D|$  integers. Then  $\prod_{i \in D} X_i Z_{N(i)}$  is in the stabilizer. Since  $XZX = -X$ , every column containing odd number of  $Z$  will contribute a  $-1$ . So total contribution when  $-1$  from columns corresponding to  $i \in D$  are multiplied will be  $(-1)^{|E(D)|}$ . This concludes the proof.  $\square$

Note that the quantum strategy for a game  $G_G$  is the same as the one used in [1]: when a player gets 1 as input, he performs an  $X$  measurement. When he gets 0 as input, he performs a  $Z$  measurement. The output is the classical measurement outcome.

An example of such games is the game defined on the complete graph  $G_{K_n}$ . The graph state  $|K_n\rangle$  is equivalent to the GHZ state on  $n$  qubits up to local transformation.

Note that for any subset  $D$  of vertices of  $K_n$ ,  $D \subseteq Even(D)$  if and only if  $|D| = 1 \mod 2$ . Now, if  $|D| = 1 \mod 2$  then the number of edges in the subgraph induced by

$D$  is  $|E(D)| = |D|(|D| - 1)/2 = (|D| - 1)/2 \bmod 2$ . Furthermore,  $|D|$  is number of 1s in the question. Hence, the loosing conditions for  $G_{K_n}$  exist when  $\sum_{j \in V} x_j = 1 \bmod 2$  and are of the form:

$$\sum_{i \in V} a_i = (\sum_{j \in V} x_j - 1)/2 + 1 \bmod 2 \quad (3)$$

This is a variation of the Mermin's parity games introduced in [22]. These games are very interesting as they have small winning probability with classical strategy[9] and their detector efficiency (see [8]) can be as low as  $1/2$  and still distinguish between classical and quantum results. The game is as follows

**Mermin's parity game:** This is a family of games for  $n$  players,  $n \geq 3$ . The task that the  $n$  players face is the following: Each player  $i$  receives as input a bit  $x_i \in \{0, 1\}$ , which is also interpreted as an integer in binary, with the promise that  $\sum_i x_i$  is divisible by 2. The players must each output a single bit  $a_i$  and the winning condition is:

$$\sum_{i=1}^n a_i = (\sum_{i=1}^n x_i)/2 \bmod 2 \quad (4)$$

$G_{K_n}$  can be transformed into Mermin's parity game using following local transformations by Player 1:  $x_1 \rightarrow x_1 + 1; a_1 \rightarrow a_1 + x_1 + 1$ . This means that given  $x_1$ , Player 1 applies above transformation on this question, plays Mermin's parity game and on the output  $a_1$  of this game applies above transformation. Same transformation works for transforming Mermin's parity game into  $G_{K_n}$ . Thus optimal winning probability is same for both games.

### 3 Genuinely $k$ -way NS nonlocality

Given a graph over  $n$  vertices, the quantum strategy that wins the corresponding graph game induces a probability distribution  $P(a_1 a_2 \dots a_n | x_1 x_2 \dots x_n)$ , where,  $x_1 x_2 \dots x_n$  are the questions and  $a_1 a_2 \dots a_n$  are the answers. In the following, we shall study the simulability of these probabilities using non-local boxes. A notation may be introduced here: given a  $D$ , if a player does not belong to  $D \cup \text{Odd}(D)$ , we represent his question with  $*$ . The  $*$  represents the case where the answer of this player is not taken into account for loosing condition.

In [1], a quantum correlation has been pointed out that cannot be simulated by any number of PR Boxes. It can easily be verified that this correlation is same as that induced by quantum strategy for graph game on  $C_5$  (cycle with 5 vertices). Our objective in rest of the article shall be to extend their argument, which we present below.

**Definition 2.** Consider players  $A_1, A_2, A_3 \dots A_n$ . Suppose the question is a string  $x_{A_1} x_{A_2} x_{A_3} \dots x_{A_n}$  with  $x_i \in 0, 1, *$ . We define the restriction of this question on a subset of players, such as  $A_1 A_2 \dots A_k$  as the string  $x_{A_1} x_{A_2} \dots x_{A_k}$ .

Now we sketch the argument given in [1]. Consider two players  $A, B$  who share a PR Box  $PR_{AB}$ . If restriction of some question  $Q$  on these players is, for example,  $1_A * B$ , then output of  $B$  is not considered for  $Q$ . So whatever protocol  $B$  follows, its effect on output of  $A$  must be same as  $B$  not using  $PR_{AB}$ . Otherwise non-signalling would be violated, and protocol of  $B$  would influence the result (by result, we mean the sum modulo 2 of outputs) of asking  $Q$  on other players. But then  $PR_{AB}$  gives a random bit that is not correlated with players involved in  $Q$ . Hence, any protocol of  $A$  that gives correct correlation for  $Q$  is equivalent to one that does not use the output of this unshared random bit generator. Hence, whenever  $A$  gets 1, we can assume that she does not use the box. If now, for example, the restriction of an another question  $Q'$  is  $1_A 1_B$ , the box  $PR_{AB}$  will be a random bit generator for  $B$ . So any protocol of  $B$  that correctly gives the correlation for  $Q'$  is equivalent to the one in which he does not use the box  $PR_{AB}$  whenever he is asked 1. Note that similar argument can be used for multipartite non-local boxes, as the crux of this argument is non-signalling.

In [2] a measure of non-locality, called *genuinely  $k$ -way NS nonlocal*, has been introduced. It can be defined in following way:

**Definition 3.** *A probability distribution induced by quantum strategy over game on a graph  $G$  is genuinely  $k$ -way NS nonlocal if this game cannot be simulated by sharing, between the players, non-local boxes that are  $k$ -partite.*

We have seen that the probability distribution obtained by the quantum strategy on game  $G_{C_5}$  cannot be simulated by two-party PR Boxes and hence is genuinely 2-way NS nonlocal. Below, we prepare grounds for an extension of this result.

**Definition 4.** *A graph  $G = (V, E)$  is  $k$ -odd dominated ( $k$ -o.d.) iff for every subset  $S \subset V$  with  $|S| = k$ , there exists a labelling of vertices in  $S$  as  $v_1, v_2, \dots, v_k$  such that there exist  $U_1, U_2, \dots, U_k$  satisfying*

- $U_i \subset V/S$
- $U_i \subseteq \text{Even}(U_i)$
- $\text{Odd}(U_i) \cap \{v_i, \dots, v_k\} = v_i$

**Lemma 3.** *Given a graph  $G(V, E)$ . If  $G$  is  $k$ -o.d., then  $G$  is  $j$ -o.d. for every  $j < k$ .*

*Proof.* We use induction and show that if  $G$  is  $k$ -o.d. then it is  $k-1$  o.d. Suppose  $G$  is  $k$ -o.d. Consider a  $S' \subset V$  satisfying  $|S'| = k-1$  and construct a subset  $S_1 = S' \cup v$ , for some  $v \in V/S'$ . Since  $G$  is  $k$ -o.d., there exists an ordering of vertices in  $S_1$  as  $v_1, v_2, \dots, v_k$ , with  $v = v_l$  for some  $l$ . Further, there exist  $U_1, U_2, \dots, U_k \subset V/S$  such that  $U_i \subseteq \text{Even}(U_i)$  and  $\text{Odd}(U_i) \cap \{v_i, \dots, v_k\} = v_i$ . Then an ordering on vertices in  $S'$  can simply be defined to be  $v_1 \dots v_{l-1} v_{l+1} \dots v_k$ . It is also easy to verify that  $\text{Odd}(U_i) \cap \{v_i, \dots, v_{l-1} v_{l+1} \dots v_k\} = v_i$ . This is true for every subset of size  $k-1$ . Hence  $G$  is  $k-1$  o.d.  $\square$

Now we have following lemma

**Lemma 4.** *Given a graph  $G(V, E)$ , if  $G$  is  $k$ -o.d. then probability distribution for quantum strategy on  $G$  is genuinely  $j$ -way NS nonlocal for all  $j \leq k$ .*

*Proof.* Lets suppose that we have a set  $S$  of size  $k$ , labelled as  $S = (v_1, v_2 \dots v_k)$ . By assumption, there exists a  $U_1 \in V/S$  such that  $Odd(U_1) \cap S = v_1$ . The question  $Q(U_1)$ , when restricted to  $S$  gives  $0_{v_1} *_{v_2} *_{v_3} \dots *_{v_k}$ . So strategy of player  $v_1$  will be equivalent to not using the box when 0 is input. Now, consider the set  $S' = S/v_1$ . There exists a  $U_2 \in V/S$  such that  $Odd(U_2) \cap S' = v_2$ . Consider  $Q(U_2)$ , which when restricted to  $S$  looks like  $*_{v_1} 0_{v_2} *_{v_3} \dots *_{v_k}$  or  $0_{v_1} 0_{v_2} *_{v_3} \dots *_{v_k}$  depending on whether  $v_1 \notin Odd(U_2)$  or  $v_1 \in Odd(U_2)$ , respectively. In either case, strategy of  $v_2$  will be equivalent to not using the box when 0 is input. Continuing this way, every strategy is equivalent to the one in which no vertex in  $S$  uses the box when 0 is input.

Now consider  $Q(v_i)$  for  $i \in (1, k)$ . When restricted to  $S$ , it is  $x_{v_1} \dots x'_{v_{i-1}} 1_{v_i} x''_{v_{i+1}} \dots x'''_{v_k}$ , where  $x, x', x'' \dots x''' \in 0, *$ . Thus,  $v_i$  plays the strategy equivalent to not using the box when 1 is input. In a similar fashion, all of the vertices in  $S$  play strategies that do not use the box when 1 is input. Since this is true for all  $S'$  of size  $k$ , probability distribution for quantum strategy on game  $G_G$  cannot be simulated by  $k$ -partite boxes. Hence this probability distribution is genuinely  $k$ -way NS nonlocal. Now, invoking Lemma 3 and using induction, we see that the probability distribution is genuinely  $j$ -way NS nonlocal for all  $j < k$ .  $\square$

We give below, as a corollary, a sufficient condition under which a graph is  $k$ -o.d. This condition shall be useful in study of games over some graphs below (even though it is more restrictive, it is easier to check compared to  $k$ -o.d. property).

**Corollary 1.** *Consider a graph  $G = (V, E)$  and a fixed  $k$ . For every  $j \leq k$ , if for every subset  $S' \subset V$  of size  $j$  there exist  $k - j + 1$  disjoint subsets  $W_1, W_2 \dots W_{k-j+1}$  in  $V \setminus S'$  each of which satisfy  $W_i \subset Even(W_i)$  and  $|Odd(W_i) \cap S'| = 1$ , then  $G$  is  $k$ -o.d.*

*Proof.* To show that a graph is  $k$ -o.d., it suffices to construct the labels  $v_1, v_2 \dots v_k$  and subsets  $U_1, U_2 \dots U_k$  as stated in Definition 3. Given a subset  $S$  of size  $k$ , let  $S' = S$ . There exists a subset  $W_1 \in V/S'$  that satisfies  $|Odd(W_1) \cap S'| = 1$  and  $W_1 \subset Even(W_1)$ . Label the vertex adjacent to  $W_1$  with  $v_1$  and set  $U_1 = W_1$ . Now, set  $S' = S/v_1$ . There exist two disjoint subsets  $W_2, W'_2 \in V/S'$  that satisfy  $|Odd(W_2) \cap S'| = |Odd(W'_2) \cap S'| = 1$  and  $W_2 \subset Even(W_2), W'_2 \subset Even(W'_2)$ . If  $W_2$  contains  $v_1$ , then label as  $v_2$  the vertex adjacent to  $W'_2$  and set  $U_2 = W'_2$ , else label as  $v_2$  the vertex adjacent to  $W_2$  and set  $U_2 = W_2$ . Continuing this way, the labels  $v_1, v_2 \dots v_k$  and the sets  $U_1, U_2 \dots U_k$  can be constructed. Hence  $G$  is  $k$ -o.d.  $\square$

### 3.1 Example of 3-o.d. and 4-o.d. graphs

The Paley graphs on  $n$  vertices ( $Pal_n$ ) are defined when  $n$  is a prime power and satisfies  $n \equiv 1 \pmod{4}$ . Vertices are labelled with numbers  $0, 1 \dots (n-1)$  and two vertices  $a, b$  are adjacent if and only if  $a - b = k^2 \pmod{n}$  for some  $k$ . These graphs have recently been investigated for their interesting properties under local complementation [20]. They

are up to our knowledge the only known family of graphs with minimal degree by local complementation larger than the square root of their order, which implies by [19] that to prepare the Paley graph states by measurement only, one needs  $\sqrt{n}$  qubits measurements. They are also the best known family for secret sharing with graph states and thus for some quantum codes [21, 17].

Furthermore, these graphs have interesting properties such as: they are *strongly regular* ( $Srg(n, (n-1)/2, (n-5)/4, (n-1)/4)$ ), *self-complementary*, and *symmetric*; that is, it is vertex-transitive and edge-transitive (see for example [5]).

Using Corollary 1 we show that every Paley graph having more than 5 vertices is 3-o.d. and that the probability distribution obtained by the game on  $|Pal_{13}\rangle$  is genuinely 4-way NS nonlocal.

**Lemma 5.** *For  $n \geq 5$  Paley graphs on  $n$  vertices satisfy the conditions of Corollary 1 with subsets  $W_i$  of cardinality 1 for  $k = 3$  and thus are 3-o.d.*

*Proof.* A graph Paley- $n$  is strongly regular: every vertex has  $(n-1)/2$  neighbours, any two adjacent vertices have  $(n-5)/4$  vertices in their common neighbourhood and any two non-adjacent vertices have  $(n-1)/4$  vertices in their common neighbourhood. Consider a subset  $S$  of size 1. There exist  $(n-1)/2$  vertices adjacent to it, which is  $\geq 3$  for  $n > 5$ . For a subset  $S = (v_1, v_2)$  of size 2, there are at most  $(n-1)/4$  vertices adjacent to both  $v_1, v_2$ . Thus there are at least  $(n-1)/2 + (n-1)/2 - (n-1)/4 - 1 = 3(n-1)/4 - 1$  vertices  $\notin S$  adjacent to exactly one vertex in  $S$ , which is  $> 2$ . A  $-1$  appear to count the possibility that  $v_1, v_2$  are adjacent. Now, for any subset  $S = (v_1, v_2, v_3)$  of size 3, there are at least  $3(n-1)/4 - 1$  vertices (we collectively call them  $J$ ) adjacent to exactly one of  $v_1, v_2$ . Suppose  $v_3 \notin J$ . Number of vertices in  $J$  adjacent to  $v_3$  are at most  $(n-1)/2$ . Hence, there are at least  $(n-1)/4 - 1$  vertices adjacent to exactly one of the vertices in  $S$ . Now, if  $v_3 \in J$ , then size of  $J/v_3$  is at least  $(n-1)/4 - 2$ . But now, number of vertices in  $J$  adjacent to  $v_3$  are at most  $(n-1)/2 - 1$ . Hence, there are at least  $(n-1)/4 - 1$  vertices adjacent to exactly one of the vertices in  $S$ , which is  $> 1$ . Proof follows by Corollary 1, with each disjoint subset  $W_i$  satisfying  $W_i \subseteq Even(W_i)$  is a vertex.  $\square$

Note that the argument in above lemma cannot be extended to subsets of size 4.

**Lemma 6.** *The probability distribution obtained by the game on  $|Pal_{13}\rangle$  (Figure 1) is genuinely 4-way NS nonlocal.*

*Proof.* For genuinely 4-way NS nonlocality in graph game over  $Pal_{13}$ , Corollary 1 takes this form: Let  $k = 4$ . For every  $j \leq 4$ , if for every subset  $S' \subset V$  of size  $j$  there exist  $5-j$  disjoint subsets  $U_1, U_2, \dots, U_{5-j}$  in  $V/S'$  each of which satisfy  $|Odd(U_i) \cap S'| = 1$  and  $U_i \subset Even(U_i)$ , then  $G$  is 4-o.d.

By Lemma 5, this corollary is satisfied for  $j = 1, 2, 3$  with  $U_i$  of size 1.

For  $j = 4$ , we verify it case by case below, whence  $U_i$  may not be subsets of size 1 in some cases, but they satisfy  $U_i \subset Even(U_i)$ .



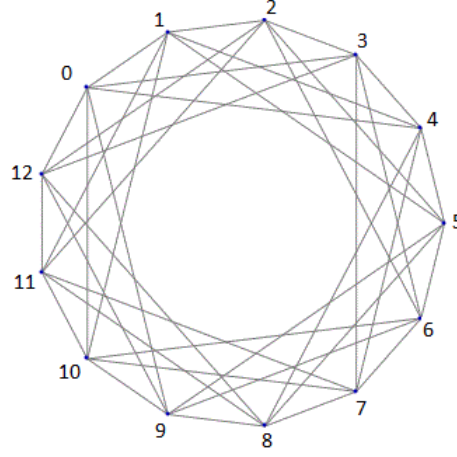


Figure 1: Paley graph on 13 vertices

By edge transitivity and self-complementarity, we can always select 1, 2 or 1, 3 as two elements of set  $S$ . Further, a subgraph isomorphic to  $K_4$  does not exist in Paley-13. So there are no set of four vertices with no edge between them, as the graph is self-complementary [26]. Our cases are as follows, classified on the basis of number of edges.

1. **Five edges:** Possibilities are: 1, 2, 5, 6 and 1, 2, 5, 11.
  - For case 1- 7 is adjacent to only 6.
  - For case 2 - 9 is adjacent to 5.
2. **Four edges:** We have following possibilities: 1, 2, 3, 4; 1, 2, 6, 10 (rectangles); 1, 2, 11, 0; 1, 2, 3, 11; 1, 2, 4, 11; 1, 2, 6, 11; 1, 2, 7, 11; 1, 2, 8, 11 (triangle and an edge).
  - Case 1- 10 is adjacent to only 1.
  - Case 2- 12 is adjacent to only 1
  - Case 3- 8 is adjacent to only 11.
  - Case 4- 8 is adjacent to only 11.
  - Case 5- 6 is adjacent to 2 only.
  - Case 6- 0 is adjacent only to 1.
  - Case 7- 0 is adjacent only to 1.
  - Case 8 - 0 is adjacent only to 1.
3. **Three edges:** Possibilities are : 1, 2, 9, 11 (triangle with an independent vertex); 1, 2, 4, 10 (star); 1, 2, 3, 7; 1, 2, 3, 12; 1, 2, 6, 7; 1, 2, 6, 9; 1, 2, 12, 8; 1, 2, 12, 9; 1, 3, 4, 12; 1, 3, 4, 6; 1, 3, 6, 10; 1, 3, 7, 10.

- Case 1- 7 is adjacent to only 11.
  - Case 2- 9 is adjacent to only 10.
  - Case 3- 8 is adjacent only to 7.
  - Case 4- 9 is adjacent to only 12.
  - Case 5- 0 is adjacent to 1 only.
  - Case 6- 8 is adjacent to only 9.
  - Case 7- 7 is adjacent to 8 only.
  - Case 8- 4 is adjacent to 1 only.
  - Case 9- 10 is adjacent to 1 only.
  - Case 10- 11 is adjacent to 1 only.
  - Case 11- 12 is adjacent to 3 only.
  - Case 12- 9 are adjacent to only 10.
4. **Two edges:** Possibilities are: 1, 2, 4, 9; 1, 3, 8, 11; 1, 3, 5, 8 (two edges forming a ray and one vertex independent); 1, 2, 7, 8; 1, 3, 6, 11; 1, 3, 5, 7 (one edge on two vertices and another edge on remaining vertices).
- Case 1- 7 is adjacent to only 4.
  - Case 2- 6 is adjacent to 3 only.
  - Case 3- 10 is adjacent to 1 only.
  - Case 4- 9 is adjacent only to 1.
  - Case 5- 9 is adjacent to only 6.
  - Case 6- 12 is adjacent to only 3.
5. **One edge:** Possibilities are: 1, 2, 9, 7; 1, 3, 9, 11; 1, 3, 8, 10. These cases have the difficulty that there is no vertex in remaining graph that is adjacent to only one of the vertices.
- Case 1- We consider  $U = (4, 6, 12)$ .  $Odd(U)$  contains only 1 among 1, 2, 7, 9.
  - Case 2- We consider  $U = (10, 8)$ .  $Odd(U)$  contains only 1.
  - Case 3- We consider  $U = (7, 9)$ .  $Odd(U)$  contains only 3.
6. **No edge:** There are no possibilities here.

□

### 3.2 $k$ -existential closure and $k$ -odd domination

Another sufficiency condition can be given, which is motivated by a more restrictive graph theoretic notion called  $k$ -existential closure. It is one of the adjacency properties that were originally introduced by Erdős, Rényi in [16] and is defined as follows :

**Definition 5.** For a fixed  $k \geq 1$ , a graph  $G$  is  $k$ -existentially closed or  $k$ -e.c. if for every  $k$ -element subset  $S$  of the vertices, and for every subset  $T$  of  $S$ , there is a vertex not in  $S$  which is joined to every vertex in  $T$  and to no vertex in  $S/T$ .

An interesting survey on known families of  $n$ -e.c. graphs, that include variants on Paley graphs, graphs related to Hadamard matrices etc., can be found in [5].

**Corollary 2.** If a graph  $G$  is  $k$ -e.c., then it is  $k$ -o.d.

*Proof.* Consider any subset  $S$  of size  $k$ . Assign any labelling to vertices and let it be  $v_1, v_2, \dots, v_k$ . Consider the subset  $v_i, \dots, v_k$  for all  $i \leq k$ . Let  $T = (v_1, v_2, \dots, v_i)$ . Then there exists a vertex  $u_i$  in  $V/S$  that is adjacent to all vertices in  $T$  and to no vertex in  $S/T$ . Further,  $u_i \in \text{Even}(u_i)$ . So set  $U_i = u_i$ . It satisfies the condition  $\text{Odd}(U_i) \cap \{v_i, \dots, v_k\} = \emptyset$ . Hence the graph is  $k$ -o.d.  $\square$

This condition allows us to sketch the nature of non-locality in many quantum probability distributions. It is known that almost all finite graphs are  $n$ -e.c [16] and hence, by above corollary, quantum probability distributions corresponding to almost all graph games are genuinely  $n$ -way NS nonlocal. Moreover, as stated in [5], using a famous result from number theory on character sums estimates (Weil's proof of the Riemann hypothesis over finite fields) [6] proved bounds for the  $n$ -e.c. property of Paley graphs which allows us to have the following general result for Paley graph states:

**Theorem 1.** For any  $n$ , the quantum probability distribution obtained with Paley graph states of size greater than  $n^2 * 2^{2n-2}$  is genuinely  $n$ -way NS nonlocal.

*Proof.* As proved in [6], a Paley graph having more than  $k^2 2^{2k-2}$  vertices is  $k$ -e.c. Thus by corollary 2 it is also  $k$ -o.d. Hence, Lemma 4 allows to conclude.  $\square$

Note that the graphs  $Pal_n$  are 3-e.c. only for  $n \geq 29$  (see [4]), thus the general result could not be applied for our discussion related to  $Pal_{13}$  in previous subsection.

Furthermore in [5] it is stated that an  $n$ -e.c. graph has at least  $n + 2^n$  vertices, thus Corollary 2 cannot be used to prove better than logarithmic bounds for genuine  $n$ -way NS nonlocality.

## 4 Conclusion

Using graph states and simple measurements, we have defined a new family of pseudo-telepathy games generalizing the Mermin game, and we have shown that the probability distribution obtained using the Paley graph states exhibits a strong multipartite nonlocality.

It would be interesting to study more thoroughly the graph games defined here, for example to find the general expression for probability of winning the games in the best classical strategy. This question shall allow us to study the behaviour of these games in reference to inefficiency in detectors, a problem addressed in general in [8].

The results in [19] show that preparing a graph state  $|G\rangle$  on  $n$  qubits with measurement only, requires measurements on  $\delta_{loc}(G) + 1$  qubits simultaneously, where  $\delta_{loc}(G)$  is the minimum degree by local complementation. Thus for graph states, the minimum degree by local complementation is a measure of multipartite nonlocality for which the complete graph (the GHZ state) behave poorly ( $\delta_{loc}(K_n) = 1$ ) and that is high for Paley graph states [20] ( $\delta_{loc}(Pal_n) \geq \sqrt{n}$ ) (it is also mentioned in [20] that the question of the existence of a subfamily of Paley graph states requiring measurements on  $c.n$  qubits for some constant  $c$  is equivalent to a known conjecture in code theory). Thus relating  $k$ -o.d. or a weaker condition for NS genuine  $n$ -way nonlocality with the minimum degree by local complementation could give better bounds for Paley graphs (in this paper we proved only genuine logarithmic NS nonlocality for Paley graph states). It would also be interesting to have a combinatorial necessary condition for genuine NS  $k$ -way nonlocality for the probability distributions obtained by the graph games.

An other interesting direction is the extension where a player can have a set of qubits (a set of vertices of the graph) both in the case where the input is binary and when the input is an integer.

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